# The Cholesky Decomposition - Part I

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June, 2012

A Cholesky matrix transforms a vector of uncorrelated (i.e. independent) normally-distributed random variates into a vector of correlated (i.e. dependent) normally-distributed random variates. These now correlated random variates can be used in a Monte Carlo simulation where correlated random variates are required. In Part I we will develop the mathematics of the Cholesky Decomposition. To develop the mathematics we will use the following hypothetical problem...

**The Problem:** Imagine that we are tasked with creating a Monte Carlo simulation of a stochastic cash flow stream where cash flow ($C_t$) for any year $t$ is defined as...

$$C_t = R_{t-1}\,(1+\theta_1)\,(1-\theta_2-\theta_3) \tag{1}$$

In the cash flow equation above $R_{t-1}$ is revenue for the prior year, $\theta_1$ is a random variate that represents the revenue growth rate, $\theta_2$ is a random variate that represents the ratio of operating expenses to revenue, and $\theta_3$ is a random variate that represents the ratio of capital expeditures to revenue. The probability distributions associated with each random variate and the three independent random variates pulled from those distributions are..

Table 1 - Probability Distributions and Three Independent Random Variates

| Symbol | Description | Distribution | Mean | Standard Deviation | Random Variate |
|---|---|---|---|---|---|
| $\theta_1$ | Revenue growth rate | Normal | 0.04 | 0.05 | (0.0370) |
| $\theta_2$ | Expenses to revenue | Normal | 0.60 | 0.15 | 0.6685 |
| $\theta_3$ | Cap ex to revenue | Normal | 0.20 | 0.07 | 0.2956 |

**Example:** Given that revenue for the prior year was \$1,000,000 simulated cash flow for the current year using cash flow Equation (1) and the three random variates from Table 1 is...

$$C_t = \$1,000,000\,(1+(0.0370))\,(1-0.6685-0.2956) = \$34,538 \tag{2}$$

The correlation matrix below represents the correlations of cash flow Equation (1) random variates...

Table 2 - Correlation Matrix

| | $\theta_1$ | $\theta_2$ | $\theta_3$ |
|---|---|---|---|
| $\theta_1$ | 1.00 | 0.35 | 0.55 |
| $\theta_2$ | 0.35 | 1.00 | 0.25 |
| $\theta_3$ | 0.55 | 0.25 | 1.00 |

**Question:** The three random variates in Table 1 are independent and therefore the Monte Carlo simulated cash flow (via Equation (2)) does not reflect the correlation matrix as defined in Table 2 above. If correlation was taken into account what would the revised result of Equation (2) be? Why is the answer different?

## A Vector Of Independent Random Variates

We will define vector $\vec{\mathbf{x}}$ to be a vector in $\mathbb{R}^3$ that consists of three independent, standardized, normally-distributed random variates. This vector in vector notation is....

$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{3}$$

Each random variate in vector $\vec{\mathbf{x}}$ is normally-distributed with mean zero and variance one. The probability distributions for each random variate in equation form are...

$$x_1 \sim N[0,1] \text{ ...and... } x_2 \sim N[0,1] \text{ ...and... } x_3 \sim N[0,1] \tag{4}$$

The expected values of each $x_i$ in vector $\vec{\mathbf{x}}$ where $i \in [1,2,3]$ are...

$$\mathbb{E}[x_1] = 0 \text{ ...and... } \mathbb{E}[x_2] = 0 \text{ ...and... } \mathbb{E}[x_3] = 0 \tag{5}$$

The expected values of the square of each $x_i$ in vector $\vec{\mathbf{x}}$ where $i \in [1,2,3]$ are...

$$\mathbb{E}[x_1^2] = 1 \text{ ...and... } \mathbb{E}[x_2^2] = 1 \text{ ...and... } \mathbb{E}[x_3^2] = 1 \tag{6}$$

Given that the random variates are independent the expected values of the product of each $x_i\, x_j$ pair in vector $\vec{\mathbf{x}}$ where $i \in [1,2,3]$, $j \in [1,2,3]$ and $i \neq j$ are...

$$\mathbb{E}[x_1 x_2] = 0 \text{ ...and... } \mathbb{E}[x_1 x_3] = 0 \text{ ...and... } \mathbb{E}[x_2 x_3] = 0 \tag{7}$$

## Adding Dependence Via A Linear Transformation

We have the linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $T(\vec{\mathbf{x}}) = \mathbf{A}\vec{\mathbf{x}}$. The matrix $\mathbf{A}$ is the transformation matrix for $T$ with respect to the standard basis and is in the following form...

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{8}$$

As noted above vector $\vec{\mathbf{x}}$ consists of **independent**, normally-distributed random variates each with mean zero and variance one. We want to apply a linear transformation to vector $\vec{\mathbf{x}}$ such that the matrix:vector product of this transformation is vector $\vec{\mathbf{y}}$, which is a vector in $\mathbb{R}^3$ that consists of **dependent**, normally-distributed random variates with mean zero and variance one. Using Equations (3) and (8) this linear transformation in equation form is...

$$\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{y}}$$
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{9}$$

If we multiply out the linear transformation as defined by Equation (9) above the equations for each **dependent** random variate $y_i$ as a function of the **independent** random variates $x_i$ are...

$$y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \tag{10}$$
$$y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \tag{11}$$
$$y_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \tag{12}$$

We will define matrix $\mathbf{C}$ to be the covariance matrix applicable to the vector of dependent random variates $\vec{\mathbf{y}}$ as defined by Equation (9) above. Using Appendix Equations (49) through (54) our covariance matrix $\mathbf{C}$ in matrix notation is...

$$\mathbf{C} = \begin{bmatrix} a_{11}^2 + a_{12}^2 + a_{13}^2 & a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} & a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} \\ a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} & a_{21}^2 + a_{22}^2 + a_{23}^2 & a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} \\ a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} & a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} & a_{31}^2 + a_{32}^2 + a_{33}^2 \end{bmatrix} \tag{13}$$

Note that the product of our linear transformation matrix $\mathbf{A}$ (as defined by Equation (8) above) and it's transpose also gives us covariance matrix $\mathbf{C}$. In equation form this relationship is...

$$\mathbf{A}\mathbf{A}^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \mathbf{C} \tag{14}$$

**Important:** When vector $\vec{\mathbf{x}}$ as defined by Equation (3) above consists of **independent**, standardized, normally-distributed random variates then per Equations (13) and (14)...

$$\mathbf{C} = \mathbf{A}\mathbf{A}^T \tag{15}$$

## The Plan

Cash flow Equation (2) resulted in simulated cash flow of \$34,538. We will define the vector $\vec{\mathbf{z}}$ to be the vector of parameters used in that equation (note that the parameters were given to us in Table 1). The definition of vector $\vec{\mathbf{z}}$ is therefore...

$$\vec{\mathbf{z}} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} (0.0370) \\ 0.6685 \\ 0.2956 \end{bmatrix} \tag{16}$$

The random variates in vector $\vec{\mathbf{z}}$ are independent, non-standardized, normally-distributed random variates. Using these parameters in cash flow Equation (2) gives us an incorrect result because the correlation matrix as defined in Table 2 is ignored. To answer our problem we must add dependence to vector $\vec{\mathbf{z}}$ consistent with our correlation matrix as defined in Table 2 and then recalculate cash flow Equation (2) such that we get the correct result.

To solve our problem we will...

| Step | Course of Action |
|---|---|
| 1 | Standardize the random variates in vector $\vec{\mathbf{z}}$. Note that standardization does not alter the correlation matrix as given to us in Table 2. |
| 2 | Noting that when using standardized random variates the covariance matrix equals the correlation matrix we set matrix $\mathbf{C}$ equal to the correlation matrix as defined in Table 2. |
| 3 | Find transformation matrix $\mathbf{A}$ by decomposing matrix $\mathbf{C}$ into matrix $\mathbf{A}$ and it's transpose via a Cholesky Decomposition. |
| 4 | Calculate the matrix:vector product of our now defined matrix $\mathbf{A}$ and our vector of independent, standardized random variates such that we get a vector of dependent, standardized random variates. |
| 5 | Convert these dependent, standardized, normally-distributed random variates with mean zero and variance one to the probability distributions of $\theta_1$, $\theta_2$ and $\theta_3$ (i.e. un-standardize). |
| 6 | Redefine vector $\vec{\mathbf{z}}$ and recalculate Equation (2) to solve our problem. |

## Step 1 - Standardize The Random Variates

We will define $x_1$ to be the standardized random variate for $\theta_1$ as defined by vector $\vec{\mathbf{z}}$ above. The standardized random variate $x_1$ is...

$$x_1 = \frac{-0.0370 - 0.04}{0.05} = -1.5404 \tag{17}$$

We will define $x_2$ to be the standardized random variate for $\theta_2$ as defined by vector $\vec{\mathbf{z}}$ above. The standardized random variate $x_2$ is...

$$x_2 = \frac{0.6685 - 0.60}{0.15} = 0.4566 \tag{18}$$

We will define $x_3$ to be the standardized random variate for $\theta_3$ as defined by vector $\vec{\mathbf{z}}$ above. The standardized random variate $x_3$ is...

$$x_3 = \frac{0.2956 - 0.20}{0.07} = 1.3664 \tag{19}$$

Our vector of standardized random variates is...

$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1.5404) \\ 0.4566 \\ 1.3664 \end{bmatrix} \tag{20}$$

Note: When you standardize a normally-distributed random variate you subtract the mean and divide by the standard deviation. That process gives you a random variate with mean zero and variance one.

## Step 2 - Define The Covariance Matrix C

As noted above standardization does change the correlation matrix as defined by Table 2 above. When random variates are standardized the covariance matrix equals the correlation matrix. Using Table 2 above we can therefore define our covariance matrix $\mathbf{C}$ as...

$$\mathbf{C} = \begin{bmatrix} 1.00 & 0.35 & 0.55 \\ 0.35 & 1.00 & 0.25 \\ 0.55 & 0.25 & 1.00 \end{bmatrix} \tag{21}$$

## Step 3 - The Cholesky Decomposition

We will define matrix $\mathbf{A}$ by decomposing matrix $\mathbf{C}$, which is now the correlation matrix as defined in Table 2 above, into matrix $\mathbf{A}$ and it's transpose. We will not only decompose matrix $\mathbf{C}$ into a lower and upper triangular matrix via a **LU Decomposition** but we will also construct the decomposition such that matrix $\mathbf{U}$ equals the transpose of matrix $\mathbf{L}$. In general the equation for this process is...

$$\begin{aligned}\mathbf{C} &= \mathbf{LU}\\ \mathbf{C} &= \tilde{\mathbf{L}}\tilde{\mathbf{L}}^T\\ \mathbf{A}\mathbf{A}^T &= \tilde{\mathbf{L}}\tilde{\mathbf{L}}^T\\ \mathbf{A} &= \tilde{\mathbf{L}}\end{aligned} \tag{22}$$

We will now work through the decomposition. Per Equation (15) above...

$$\begin{aligned}\mathbf{A}\mathbf{A}^T &= \mathbf{C}\\ &= \begin{bmatrix}1.00 & 0.35 & 0.55\\ 0.35 & 1.00 & 0.25\\ 0.55 & 0.25 & 1.00\end{bmatrix}\end{aligned} \tag{23}$$

To define matrix $\mathbf{A}$ we will perform an LU decomposition of matrix $\mathbf{C}$. We will first find the upper triangular matrix $\mathbf{U}$. If...

$$\mathbf{E_1} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ (0.55) & 0 & 1\end{bmatrix} \text{ ...and... } \mathbf{E_2} = \begin{bmatrix}1 & 0 & 0\\ (0.35) & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \text{ ...and... } \mathbf{E_3} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & (0.0655) & 1\end{bmatrix} \tag{24}$$

Then our upper triangular matrix $\mathbf{U}$ is...

$$\mathbf{U} = \mathbf{E_3}\mathbf{E_2}\mathbf{E_1}\mathbf{C} = \begin{bmatrix}1 & 0.3500 & 0.5500\\ 0 & 0.8775 & 0.0575\\ 0 & 0 & 0.6937\end{bmatrix} \tag{25}$$

We will then find the lower triangular matrix $\mathbf{L}$. Given that...

$$\mathbf{E_1^{-1}} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0.55 & 0 & 1\end{bmatrix} \text{ ...and... } \mathbf{E_2^{-2}} = \begin{bmatrix}1 & 0 & 0\\ 0.35 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \text{ ...and... } \mathbf{E_3^{-3}} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0.0665 & 1\end{bmatrix} \tag{26}$$

Then our lower triangular matrix $\mathbf{L}$ is...

$$\mathbf{L} = \mathbf{E_1^{-1}}\mathbf{E_2^{-1}}\mathbf{E_3^{-1}} = \begin{bmatrix}1 & 0 & 0\\ 0.3500 & 1 & 0\\ 0.5500 & 0.0665 & 1\end{bmatrix} \tag{27}$$

If we redefine the upper triangular matrix $\mathbf{U}$ as...

$$\mathbf{U} = \mathbf{D}\mathbf{L^T} \tag{28}$$

Then...

$$\mathbf{D} = \mathbf{U}\Big[\mathbf{L^T}\Big]^{-1} = \begin{bmatrix}1 & 0.3500 & 0.5500\\ 0 & 0.8775 & 0.0575\\ 0 & 0 & 0.6937\end{bmatrix}\begin{bmatrix}1 & (0.3500) & (0.5271)\\ 0 & 1 & (0.0655)\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 0.8775 & 0\\ 0 & 0 & 0.6937\end{bmatrix} \tag{29}$$

The square root of matrix D is...

$$\mathbf{D^{0.5}} = \begin{bmatrix}1 & 0 & 0\\ 0 & 0.9367 & 0\\ 0 & 0 & 0.8329\end{bmatrix} \text{ ...where... } \Big[\mathbf{D^{0.5}}\Big]^T = \begin{bmatrix}1 & 0 & 0\\ 0 & 0.9367 & 0\\ 0 & 0 & 0.8329\end{bmatrix} = \mathbf{D^{0.5}} \tag{30}$$

Putting it all together...

$$\begin{aligned}\mathbf{C} &= \mathbf{LU} \\ &= \mathbf{LDL^T} \\ &= \mathbf{LD^{0.5}D^{0.5}L^T} \\ &= \mathbf{LD^{0.5}}[\mathbf{D^{0.5}}]^T\mathbf{L^T} \\ &= \mathbf{LD^{0.5}}[\mathbf{LD^{0.5}}]^T \end{aligned} \tag{31}$$

So matrix $\mathbf{A}$ (also known as the Cholesky matrix) is...

$$\mathbf{A} = \mathbf{LD^{0.5}} = \begin{bmatrix} 1 & 0 & 0 \\ 0.3500 & 1 & 0 \\ 0.5500 & 0.0665 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.9367 & 0 \\ 0 & 0 & 0.8329 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.3500 & 0.9367 & 0 \\ 0.5500 & 0.0614 & 0.8329 \end{bmatrix} \tag{32}$$

We have found our linear transformation matrix $\mathbf{A}$!

## Step 4 - Calculate Our Vector Of Dependent, Standardized Random Variates

We will define vector $\vec{\mathbf{y}}$ to be a vector of dependent, standardized, normally-distributed random variates. Given our linear transformation matrix $\mathbf{A}$ as defined by Equation (32) above, and vector $\vec{\mathbf{x}}$ as defined by Equation (20) above, our vector of dependent, standardized, normally-distributed random variates is...

$$\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{y}}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.3500 & 0.9367 & 0 \\ 0.5500 & 0.0614 & 0.8329 \end{bmatrix} \begin{bmatrix} (1.5404) \\ 0.4566 \\ 1.3664 \end{bmatrix} = \begin{bmatrix} (1.5404) \\ (0.1114) \\ 0.3189 \end{bmatrix} \tag{33}$$

## Step 5 - Recalculate The Parameters To Our Problem

Vector $\vec{\mathbf{y}}$ as defined by Equation (33) above consists of dependent, standardized, normally-distributed random variates. The parameters to our problem $\theta_1$, $\theta_2$ and $\theta_3$ are not standardized (i.e. have means and/or variances different from zero and one, respectively). Per Equation (33) above vector $\vec{\mathbf{y}}$ is defined as...

$$y_1 = -1.5404 \;\; \textit{...and...} \;\; y_2 = -0.1114 \;\; \textit{...and...} \;\; y_2 = 0.3189 \tag{34}$$

Our new value of parameter $\theta_1$, which is $y_1$ un-standardized, is...

$$\theta_1 = 0.04 + (-1.5404)(0.05) = -0.0370 \tag{35}$$

Our new value of parameter $\theta_2$, which is $y_2$ un-standardized, is...

$$\theta_2 = 0.60 + (-0.1114)(0.15) = 0.5833 \tag{36}$$

Our new value of parameter $\theta_3$, which is $y_3$ un-standardized, is...

$$\theta_3 = 0.20 + (0.3189)(0.07) = 0.2223 \tag{37}$$

Note: To un-standardize you reverse the process of standardization. The un-standardized random variate is the mean plus the standard deviation times the standardized random variate.

## Step 6 - The Answer To Our Problem

Using our new parameters $\theta_1$, as defined by Equation (35) above, $\theta_2$, as defined by Equation (36) above, and $\theta_3$, as defined by Equation (37) above, our re-defined vector $\vec{\mathbf{z}}$ is...

$$\vec{\mathbf{z}} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} (0.0370) \\ 0.5833 \\ 0.2223 \end{bmatrix} \tag{38}$$

Plugging these new parameters into Equation (2) our revised simulated cash flow is...

$$C_t = \$1,000,000\,(1 + (0.0370))\,(1 - 0.5833 - 0.2223) = \$187,197 \tag{39}$$

Why are the answers different? Using independent random variates simulated cash flow was \$34,538. Using dependent random variates simulated cash flow was \$187,197. Note that per Table 1 the revenue growth rate $\theta_1$ was expected to be 0.0400 but was instead -0.0370 for this simulation. Per Table 2 above both $\theta_2$, which is the ratio of operating expenses to revenue, and $\theta_3$, which is the ratio of capital expenditures to revenue, are correlated with revenue growth. Given the correlation matrix as revenue growth was negative both operating expenses and capital expenditures as a percent of revenue decreased.

**THE POINT:** Correlation is important and not modeling it gives us wrong answers!

## Appendix

**A**) Using Equation (5) above the expected value of $y_1$ as defined by Equation (10) is...

$$\begin{aligned}\mathbb{E}[y_1] &= \mathbb{E}[a_{11}\,x_1 + a_{12}\,x_2 + a_{13}\,x_3]\\ &= a_{11}\,\mathbb{E}[x_1] + a_{12}\,\mathbb{E}[x_2] + a_{13}\,\mathbb{E}[x_3]\\ &= 0\end{aligned} \tag{40}$$

Using the mathematics for the expected value of $y_1$ it can be shown that the expected values of $y_2$ and $y_3$ as defined by Equations (11) and (12), respectively, are...

$$\mathbb{E}[y_2] = 0 \tag{41}$$

$$\mathbb{E}[y_3] = 0 \tag{42}$$

**B**) Using Equations (5) and (6) above the expected value of the square of $y_1$ as defined by Equation (10) is...

$$\begin{aligned}\mathbb{E}[y_1^2] &= \mathbb{E}[(a_{11}x_1 + a_{12}x_2 + a_{13}x_3)^2]\\ &= \mathbb{E}[a_{11}^2x_1^2 + a_{12}^2x_2^2 + a_{13}^2x_3^2 + 2a_{11}a_{12}x_1x_2 + 2a_{11}a_{13}x_1x_3 + 2a_{12}a_{13}x_2x_3]\\ &= a_{11}^2\mathbb{E}[x_1^2] + a_{12}^2\mathbb{E}[x_2^2] + a_{13}^2\mathbb{E}[x_3^2] + 2a_{11}a_{12}\mathbb{E}[x_1x_2] + 2a_{11}a_{13}\mathbb{E}[x_1x_3] + 2a_{12}a_{13}\mathbb{E}[x_2x_3]\\ &= a_{11}^2 + a_{12}^2 + a_{13}^2\end{aligned} \tag{43}$$

Using the mathematics for the expected value of the square of $y_1$ it can be shown that the expected values of the squares of $y_2$ and $y_3$ as defined by Equations (11) and (12), respectively, are...

$$\mathbb{E}[y_2^2] = a_{21}^2 + a_{22}^2 + a_{23}^2 \tag{44}$$

$$\mathbb{E}[y_3^2] = a_{31}^2 + a_{32}^2 + a_{33}^2 \tag{45}$$

**C**) Using Equations (6) and (7) above the expected value of the product of $y_1$ and $y_2$ as defined by Equations (10) and (11), respectively, is...

$$\begin{aligned}\mathbb{E}[y_1y_2] &= \mathbb{E}[(a_{11}x_1 + a_{12}x_2 + a_{13}x_3)(a_{21}x_1 + a_{22}x_2 + a_{23}x_3)]\\ &= \mathbb{E}[a_{11}a_{21}x_1^2 + a_{11}a_{22}x_1x_2 + a_{11}a_{23}x_1x_3 + a_{12}a_{21}x_2x_1 + a_{12}a_{22}x_2^2 +\\ &\quad a_{12}a_{23}x_2x_3 + a_{13}a_{22}x_2x_3 + a_{13}a_{23}x_3^2]\\ &= a_{11}a_{21}\mathbb{E}[x_1^2] + a_{11}a_{22}\mathbb{E}[x_1x_2] + a_{11}a_{23}\mathbb{E}[x_1x_3] + a_{12}a_{21}\mathbb{E}[x_2x_1] + a_{12}a_{22}\mathbb{E}[x_2^2] +\\ &\quad a_{12}a_{23}\mathbb{E}[x_2x_3] + a_{13}a_{22}\mathbb{E}[x_2x_3] + a_{13}a_{23}\mathbb{E}[x_3^2]\\ &= a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23}\end{aligned} \tag{46}$$

Using the mathematics for the expected value of the product of $y_1$ and $y_2$ it can be shown that the expected value of the product of $y_1$ and $y_3$ (as defined by Equations (10) and (12), respectively) and the expected value of the product of $y_2$ and $y_3$ (as defined by Equations (11) and (12), respectively) are...

$$\mathbb{E}[y_1y_3] = a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} \tag{47}$$

$$\mathbb{E}[y_2y_3] = a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} \tag{48}$$

**D**) Using Appendix Equations (40) and (43) the variance of $y_1$ is...

$$\mathbb{E}[y_1^2] - (\mathbb{E}[y_1])^2 = a_{11}^2 + a_{12}^2 + a_{13}^2 \tag{49}$$

**E**) Using Appendix Equations (41) and (44) the variance of $y_2$ is...

$$\mathbb{E}[y_2^2] - (\mathbb{E}[y_2])^2 = a_{21}^2 + a_{22}^2 + a_{23}^2 \tag{50}$$

**F**) Using Appendix Equations (42) and (45) the variance of $y_3$ is...

$$\mathbb{E}[y_3^2] - (\mathbb{E}[y_3])^2 = a_{31}^2 + a_{32}^2 + a_{33}^2 \tag{51}$$

**G**) Using Appendix Equations (46), (40) and (41) the covariance of $y_1$ and $y_2$ is...

$$\mathbb{E}[y_1 y_2] - \mathbb{E}[y_1]\mathbb{E}[y_2] = a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} \tag{52}$$

**H**) Using Appendix Equations (47), (40) and (42) the covariance of $y_1$ and $y_3$ is...

$$\mathbb{E}[y_1 y_3] - \mathbb{E}[y_1]\mathbb{E}[y_3] = a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} \tag{53}$$

**I**) Using Appendix Equations (48), (41) and (42) the covariance of $y_2$ and $y_3$ is...

$$\mathbb{E}[y_2 y_3] - \mathbb{E}[y_2]\mathbb{E}[y_3] = a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} \tag{54}$$